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LETTER TO THE EDITOR

Hyperbolic cantori have dimension zero

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Abstract. In this letter it is proved that cantori for area-preserving twist maps have dimension zero, only finitely many orbits of gaps, and gap widths go to zero exponentially in each orbit, if they are hyperbolic.

Li and Bak (1986) observed numerically that cantori for area-preserving twist maps appear to have zero measure when projected onto the angle coordinate, and furthermore to have dimension zero. In appendix 2 of MacKay *et al* (1987), it was proved that all hyperbolic cantori have projected measure zero, based on ideas of Aubry *et al* (1982).

In this letter it is proved that hyperbolic cantori have dimension zero (both Hausdorff dimension and capacity). Two proofs are given. In the first proof it is shown firstly that they have only finitely many orbits of gaps and secondly that the gap lengths go to zero exponentially in any orbit of gaps. Dimension zero follows from these results and the projected measure being zero and these results. The second proof is based on a general relation of Young (1982) between dimension, entropy and characteristic exponents of invariant measures for two-dimensional maps. Projected measure zero also follows directly from this.

One interesting open question is: what is the dimension of the union of all cantori?

For basic results on cantori and area-preserving twist maps, see Aubry and Le Daeron (1983) or MacKay and Stark (1985).

The gaps in a cantorus come in orbits, which I will label by i. Denote the angle coordinates of the endpoints of the nth gap of the ith orbit by (l_n^i, r_n^i). Choose $n=0$ to correspond to a largest gap in each orbit. Let

$$w_n^i = r_n^i - l_n^i \qquad \text{and} \qquad \varepsilon_n^i = w_n^i / w_0^i$$

so

$$\varepsilon_0^i = 1 \qquad \text{and} \qquad 0 < \varepsilon_n^i \leq 1 \qquad \text{for all } n.$$

Suppose there are infinitely many orbits of gaps. Then the sequence $(l_0^i, l_1^i, \varepsilon_1^i)$, $i \in \mathbb{N}$, is bounded in $\mathbb{R}^3/T$, identifying points under

$$T(l_0, l_1, \varepsilon_1) = (l_0 + 1, l_1 + 1, \varepsilon_1)$$

since $\operatorname{floor}(\omega) \leq l_1^i - l_0^i \leq \operatorname{ceil}(\omega)$ (the integers below and above the rotation number ω) and $\varepsilon_1^i \in [0, 1]$. Take a limit point $(x_0, x_1, \varepsilon_1)$. Generate the orbit (x_n) from (x_0, x_1). It lies on the cantorus as the cantorus is closed. Generate an orbit (ε_n) of tangent vectors to (x_n) from $(\varepsilon_0, \varepsilon_1)$ with $\varepsilon_0 = 1$. Now $\Sigma\, w_0^i < 1$, so $w_0^i \to 0$. Thus since $(x_0, x_1, \varepsilon_1)$ is a limit point of $(l_0^i, l_1^i, \varepsilon_1^i)$, given $\eta > 0$, $N > 0$, there exists i such that $|\varepsilon_n - \varepsilon_n^i| < \eta$, for all n with $|n| < N$. Hence (ε_n) is bounded in both directions in time, contradicting the assumption of hyperbolicity which would imply that every non-zero tangent orbit must grow in at least one direction of time. Thus hyperbolic cantori have only a finite number of orbits of gaps.

For a hyperbolic cantorus, the gap widths go to zero exponentially in each orbit of gaps. This follows from a standard result on hyperbolic systems (e.g. theorem 6.2 of Shub (1987)), namely, if Λ is a hyperbolic set for a diffeomorphism f, with contraction constant $\alpha < 1$, and the orbits of two points $x, y \in \Lambda$ converge together in forward time, then

$$d(f^n x, f^n y) \leqslant \alpha^n d(x, y) \qquad \text{for } n > 0$$

in an adapted metric d. The orbits of the endpoints of any gap in a cantorus converge together, hence they converge together exponentially.

Since the projected measure of a hyperbolic cantorus is zero and the gaps go to zero exponentially in each orbit of gaps at least like α^n and there are only finitely many orbits of gaps, the projected measure remaining after removing the n largest gaps is at most $C\alpha^n$, for some C. So one can cover $\pi_1(M_\omega)$ (where $\pi_1(x, y) = x$ and M_ω is the cantorus) with n intervals I_i of length $C\alpha^{-n}$. Thus for all $s > 0$, the sum

$$K_n(s) = \sum_i |I_i|^s \leqslant n\alpha^{ns}.$$

So the Hausdorff dimension, which satisfies $\mathrm{HD}(\pi_1(M_\omega)) \leqslant \inf\{s \geqslant 0\colon\ K_n(s) \to 0 \text{ as } n \to \infty\}$, is zero.

Similarly, its capacity, defined by

$$C = \limsup_{\varepsilon \to 0} \log N(\varepsilon)/\log(1/\varepsilon)$$

where $N(\varepsilon)$ is the minimum number of ε balls needed to cover the set, satisfies

$$C(\pi_1(M_\omega)) \leqslant \limsup_{n \to \infty} \frac{\log n}{\log 1/(C\alpha^n)} = 0.$$

M_ω is a Lipschitz graph over x, so has the same dimension and capacity as $\pi_1(M_\omega)$.

This result can be derived much more simply from a general result of Young (1982), namely, given a $C^{1+\beta}$, $\beta > 0$, diffeomorphism f of a surface with an ergodic invariant measure μ then

$$\mathrm{HD}(\mu) = h_\mu(f)(1/\lambda_1 + 1/|\lambda_2|)$$

provided the right-hand side is not 0/0, where

$$\mathrm{HD}(\mu) = \inf_{\mu(Y)=1} \mathrm{HD}(Y)$$

is called the Hausdorff dimension of the measure, $h_\mu(f)$ is the entropy of (f, μ) and $\lambda_1 \geqslant \lambda_2$ are the characteristic exponents of (f, μ).

A cantorus M_ω supports a unique invariant measure μ, and $h_\mu(f) = 0$, since the motion is basically an irrational rotation (e.g. Walters 1982). Also by area preservation, $\lambda_1 + \lambda_2 = 0$. Thus if $\lambda_1, \lambda_2 \neq 0$ then $\mathrm{HD}(\mu) = 0$. But for all subsets Y

$$\mathrm{HD}(Y) = \mathrm{HD}\left(\bigcup_n f^n Y\right)$$

since it is a countable union. Since the invariant measure is unique, $\bigcup f^n Y = M_\omega$ for any set Y with $\mu(Y) > 0$. Thus $\mathrm{HD}(M_\omega) = 0$.

Incidentally, from this we can easily deduce that the Lebesgue measure λ of the projection of M_ω is zero: $\mathrm{HD}(M_\omega) = 0$ implies $\mathrm{HD}(\pi_1(M_\omega)) = 0$ implies $\lambda(\pi_1(M_\omega)) = 0$.

Young also showed that the capacity of the measure, defined by

$$C(\mu) = \sup_{\delta > 0} \inf_{\mu(Y) \geqslant 1-\delta} C(Y)$$

is given by the same formula. From this we can deduce that M_ω has capacity zero, as follows. M_ω is the support of μ so, given $\varepsilon > 0$, there is a $\delta > 0$ such that $\mu(Y) \geqslant 1 - \delta$ implies that x is within ε of Y for all $x \in M_\omega$. Since $C(\mu) = 0$, given $d > 0$, there is an ε_c and a subset Y with $\mu(Y) \geqslant 1 - \delta$ which for all $\varepsilon < \varepsilon_c$ can be covered by ε^d ε balls; double the size of each ball to get a cover of M_ω by ε^d 2ε balls. Hence $C(M_\omega) \leqslant d$.

I would like to thank Peter Veerman for listening to these ideas and encouraging me to write them up, David Rand for his comments, and for pointing out the paper of Young to me, as I knew only about the results of Ledrappier (1981) and Manning (1981) before, and Henri Epstein for discussions on hyperbolicity.

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